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Numerical Experiments on Billiards

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We investigate decay properties of correlation functions in a class of chaotic billiards. First we consider the statistics of Poincaré recurrences (induced by a partition of the billiard): the results are in agreement with theoretical bounds by Bunimovich, Sinai, and Bleher, and are consistent with a purely exponential decay of correlations out of marginality. We then turn to the analysis of the velocity–velocity correlation function: except for intermittent situations, the decay is purely exponential, and the decay rates scale in a simple way with the (uniform) curvature of the dispersing arcs. A power-law decay is instead observed when the system is equivalent to an infinite-horizon Lorentz gas. Comments are given on the behaviour of other types of correlation functions, whose decay, during the observed time scale, appears slower than exponential.

KEY WORDS: Billiards; decay of correlations, scaling.

1. INTRODUCTION

In this paper we present some numerical experiments performed on a class of two-dimensional chaotic billiards. In particular we will be concerned with decay properties of correlation functions: though extensive investigations have been carried out both from a rigorous point of view[1,2] and by numerical investigations,[3–6] considerable effort is still being devoted to this problem (as regards both accurate simulations[7] and new rigorous results[8]).

We will consider both diamond (D) and Sinai billiards (S) (unit cells of a periodic Lorentz gas), see Fig. 1 and 2): the dynamics refers to a point particle with unit velocity bouncing elastically against the boundary (flat segments in S are to be considered as coinciding with a 2D torus boundary): S^t denotes the continuous time evolution, acting on the phase space $\mathcal{M}$, which is given by the set of configuration coordinates and the

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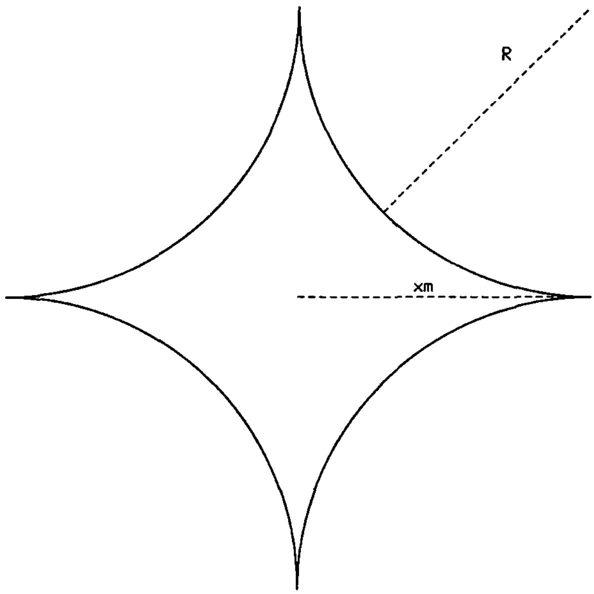

Fig. 1. Boundary of the diamond billiard system: R is the radius of curvature of the arcs, xm is taken equal to one in simulations.

unit circle of angles ω formed by $\mathbf{v}$ with a fixed direction. The system is ergodic and mixing[9] and the invariant measure is proportional to the Lebesgue measure on $\mathcal{M}$, $d\mu(z) = (2\pi A)^{-1}\, dx\, dy\, d\omega$ (A being the area of the billiard region). By considering successive collisions with the arcs we may also introduce a discrete-time dynamical system, with evolution operator T^n, on the phase space $\mathcal{M}_1$ (consisting of a coordinate l along the set of curve arcs, and $\varphi \in [-\pi/2, \pi/2]$, the angle between the incoming velocity and the outward normal of the arc at the point corresponding to l). The invariant measure for this system is $d\nu(\hat{z}) = (2P)^{-1}\, dl \cos\varphi\, d\varphi$, where P is the total arc length.

The mixing property guarantees that correlation functions vanish asymptotically: the goal here is to characterize their decay: as correlation functions are intimately linked to transport coefficients, this is an issue of the utmost physical import. For dynamical functions on the phase space correlation functions are defined as (since we are dealing with ergodic systems)

$$C_f(t) = \int_{\mathcal{M}} d\mu(z)\, f(S^t z)\, f(z) - \left(\int_{\mathcal{M}} d\mu(z)\, f(z) \right)^2 \tag{1}$$

or, when discrete dynamics is considered,

$$C_g(n) = \int_{\mathcal{M}_1} d\nu(\hat{z})\, g(T^n \hat{z})\, g(\hat{z}) - \left(\int_{\mathcal{M}_1} d\nu(\hat{z})\, g(\hat{z}) \right)^2 \tag{2}$$

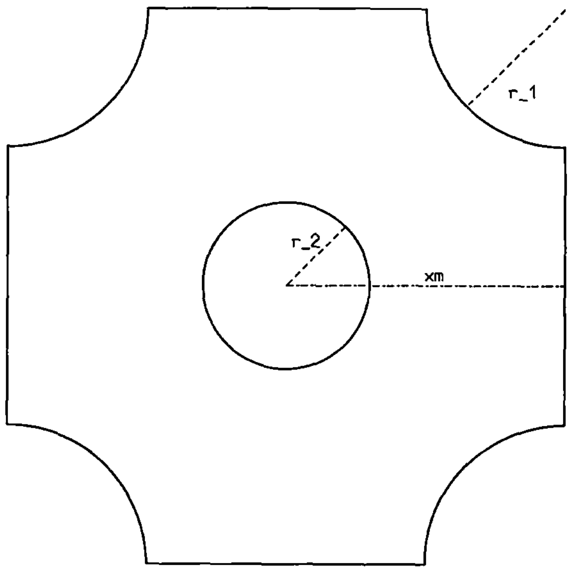

Fig. 2. Boundary of the Sinai billiard system: r_1 and r_2 are the radii of curvature of the arcs: depending on the values of geometric parameters, this system may correspond to an elementary cell of either the 0-H or the ∞-H Lorentz gas.

From a physical point a view a natural choice for the dynamical function is a component of the velocity, as it is directly related to diffusive properties of the system: for a system S corresponding to the Lorentz gas with finite horizon (henceforth called 0-H, according to ref. 7), a bound was proposed in ref. 2 in terms of a stretched exponential

$$|C_f(t)| \leqslant \exp(\alpha t^{\gamma}) \tag{3}$$

with $0 \leqslant \gamma \leqslant 1$, while in the case of discrete dynamics, a bound was obtained in ref. 1 (and refined in ref. 10)

$$|C_g(n)| \leqslant \exp(\alpha_1 n^{\gamma_1}) \tag{4}$$

with $1/2 \leqslant \gamma_1 \leqslant 1$.[10] The same bounds are supposed to hold for D, provided arcs do not meet tangentially.[2] When arcs meet tangentially, a renormalization argument[3] (which takes into account successive collisions within the same corner) strongly suggests the behavior

$$|C_g(n)| \sim \frac{1}{n} \tag{5}$$

while for the corresponding continuous dynamics a (sub)exponential bound should still hold,[2]

$$|C_f(t)| \leqslant \exp(\alpha t^{\gamma}) \tag{6}$$

In the case of a Sinai billiard corresponding to a Lorentz gas with infinite horizon (∞-H), it is believed[11,12] that

$$C_f(t) \sim \frac{1}{t} \tag{7}$$

while the discrete dynamics is still bounded exponentially, as in (4).[10] The main mathematical difficulty in dealing with these systems is due to singularities (induced by orbits touching tangentially an arc): this is reflected in the fact that Markov partitions are not finite (but countable[13,10]) and this does not ensure a purely exponential decay of correlations, as guaranteed in the finite case by standard finite-state Markov chains arguments.[14] The topological complexity of the system is also evidenced by the relatively poor performance of periodic orbit calculations,[15,16] which usually provide a very efficient tool to deal with hyperbolic systems.[17]

However, for a class of somehow simpler systems (piecewise linear automorphisms on the two-torus), Chernov[18] proved pure exponential decay of correlations in the presence of both hyperbolicity and singularities. We mention that Liverani has recently developed a different technique[8,19] which might be applicable to cases under consideration (when intermittency phenomena are not present), and is capable of proving pure exponential decay without making use of Markov partitions. It essentially probes the gap between the probability conservation eigenvalue of the Perron–Frobenius operator and the rest of the spectrum (which guides correlation decay[20–22]), by selecting an invariant cone of functions over which the Perron–Frobenius operator is a contraction according to a suitable metric: the rate of contraction gives the exponential decay rate of correlations.

For D and 0-H systems accurate numerical experiments are consistent with a pure exponential decay[7] (see also ref. 6), while earlier investigations seemed to support subexponential decay[5] (see, however, the next sections for comments on this discrepancy).

The main difficulty in numerical investigations is that when decay is very fast, statistical errors become relevant after a very short time. In particular phase averages appearing in correlation functions are estimated through Monte Carlo integration: according to (nonrigorous) conventional wisdom[3]

$$\int dV\, f \approx V\langle f\rangle \pm \left(\frac{\langle f^2\rangle - \langle f\rangle^2}{N}\right)^{1/2} \tag{8}$$

so that errors scale like $N^{-1/2}$ (N being the number of sampling points).

A subtler issue in the interpretation of numerical results is connected with sensitive dependence upon initial conditions (as we are dealing with hyperbolic systems): a positive Lyapunov exponent induces errors growing as $\delta x_0 \cdot e^{\lambda t}$: according to ref. 7, when this error (δx_0 being determined by computer precision) surpasses the statistical errors, then the averages are to be interpreted as hydrodynamic averages: nevertheless we do not observe any dramatic qualitative or quantitative change in simulations once the theoretical Lyapunov barrier is surpassed. A similar heuristic argument characterizing the role of Lyapunov-induced errors has been discussed by Ruelle,[20] when considering the behavior of correlations in the presence of noise (in numerical experiments the level of noise ν is determined by the significance employed). It is suggested that the noise will swamp the real time evolution after times of the order of $|\log \nu|/\lambda_1$, with λ_1 being the largest characteristic exponent. If correlations are investigated beyond this range, the effect of noise should lead to a pure exponential decay, with a rate coinciding with the sum of positive characteristic exponents (we never observed such a behavior in our investigations). Thus, on strictly empirical grounds, we consider the statistical errors as our more troublesome source of errors. We also remark that, according to refs. 8, 21, and 22, optimal estimates of correlation decay rates should be robust with respect to noise (within the formerly mentioned bounds), and this theoretical expectation is physically relevant, as our ultimate hope is to show the experimental relevance of our results: this implies some notion of "robustness", which should be connected to insensitivity of small error propagation, in the same fashion as for Kolmogorov's notion of "physical" invariant measures.[24]

The paper is organized as follows: we investigate preliminarily "crossing" statistics, as they are easier to access numerically: our main purpose is to show that though operating in a heuristic way, marginal behavior is correctly reproduced, and in purely hyperbolic situations pure exponential behavior is observed.

Then we turn to an analysis of the velocity–velocity correlation function. In the D case (out of marginal stability) pure exponential decay is observed, and some evidence for a scaling behavior with respect to geometrical parameter is presented (some of the results have been presented in ref. 26). We moreover show how asymptotic power-law decay seems to take place when ∞-H cases are considered.

Finally we also show results from simulations employing another class of correlation functions[5]: the qualitative features look different, yet a clear theoretical argument supporting our numerical results is still missing.

Our findings are consistent with the notion that correlation decay is governed both by generic properties of the underlying dynamical system and by the "degree of smoothness" of the dynamical functions considered.

In fact, when operating with some appropnate space of smooth functions, the decay properties are dictated by resonances[22] whose position (which rules the exponential rate of decay as well as periodic modulations) does not depend on the particular function chosen (only prefactors do). On the other hand, spectral properties of the transfer operator (which is intimately connected to resonances) are sensitive to the choice of the function space for dynamical variables (see, for example, ref. 25). Results consistent with this picture were also obtained by Crawford and Cary,[27] who investigated the decay of correlations for square-integrable functions under the action of cat maps: by introducing an appropriate choice for basis vectors, they were able to prove the existence of widely different decay laws, ranging from faster than exponential to algebraic with arbitrary exponents, depending on the smoothness of the functions considered.

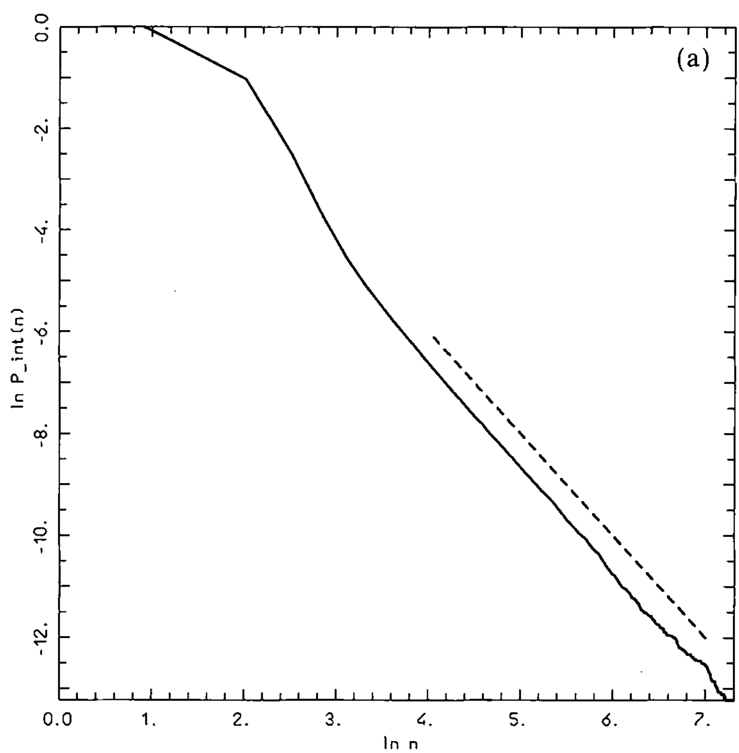

Fig. 3. (a) ln $P_{\rm int}(n)$ vs. ln n for the D tangent case ($N_{\rm cross} = 10^7$),the dashed line has slope 2. (b) ln $P_{\rm int}(t)$ vs. t for the same case; here the decay is exponential. (c) ln $P_{\rm int}(n)$ vs. n for an ∞-H case ($xm = 1$, $r_1 = r_2 = 0.2$, $N_{\rm cross} = 10^7$); again exponential behavior is found. (d) ln $P_{\rm int}(t)$ vs. ln t for the same case; a transition to power-law decay takes place: the dashed line has slope 2.

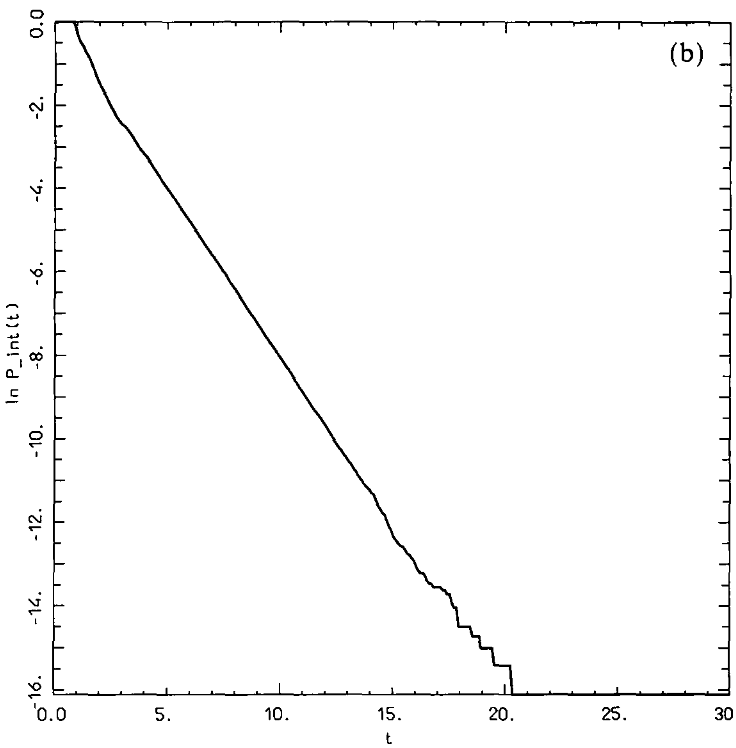

Fig. 3 (*continued*)

2. "CROSSING" STATISTICS

The problem we address is the following: we partition the configuration space of the system under investigation (using a segment with shortest length joining two facing arcs in D, or a segment through the center parallel to the torus boundary in S) and run a single (or sample of) trajectory, recording $\{t_\alpha\}$ (time from one crossing to the next) and $\{n_\alpha\}$ (number of collisions from one crossing to the next). From this data set we can approximately reconstruct $\wp(t)$ and $\wp(n)$, probability distribution functions for the crossing time or collision number; then we build the survival probabilities

$$P_{\mathrm{int}}(t) = \int_t^\infty \wp(\tau)\, d\tau, \qquad P_{\mathrm{int}}(n) = \sum_{k=n}^{\infty} \wp(k) \tag{9}$$

These are monotonically decreasing functions of their argument and they are normalized to 1 at the origin.

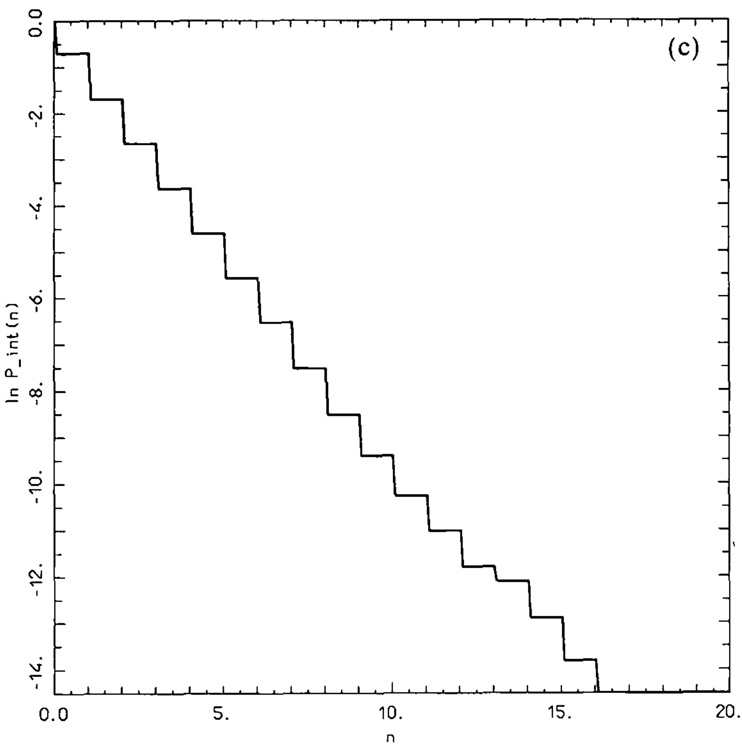

Fig. 3 (*continued*)

Our interest in these quantities originates from the fact that their integral plays a role analogous to correlation functions: it has been argued by Karney[28] (see also ref. 29) that the quantities

$$C_\tau = \int_\tau^\infty dz\, P_{\text{int}}(z), \qquad C_m = \sum_{k=m}^{\infty} P_{\text{int}}(k) \tag{10}$$

are proportional to the probability that a particle is trapped in the same region at two times τ (or m) apart: an analogous reasoning has been invoked by ref. 30 in dealing with the transition to chaos in standard-like mappings.

The argument[28] goes roughly as follows (we consider the case of discrete dynamics): if we denote by n_c the mean collision number, then $\wp(k)/n_c$ is the probability that an iterate chosen at random is the endpoint of a string of k iterates on the same region, while $k\wp(k)/n_c$ is the probability that an iterate chosen at random belongs to a string with collision time equal to k. So the probability that two iterates τ apart chosen at random belong to the same string of single-subdomain collisions is

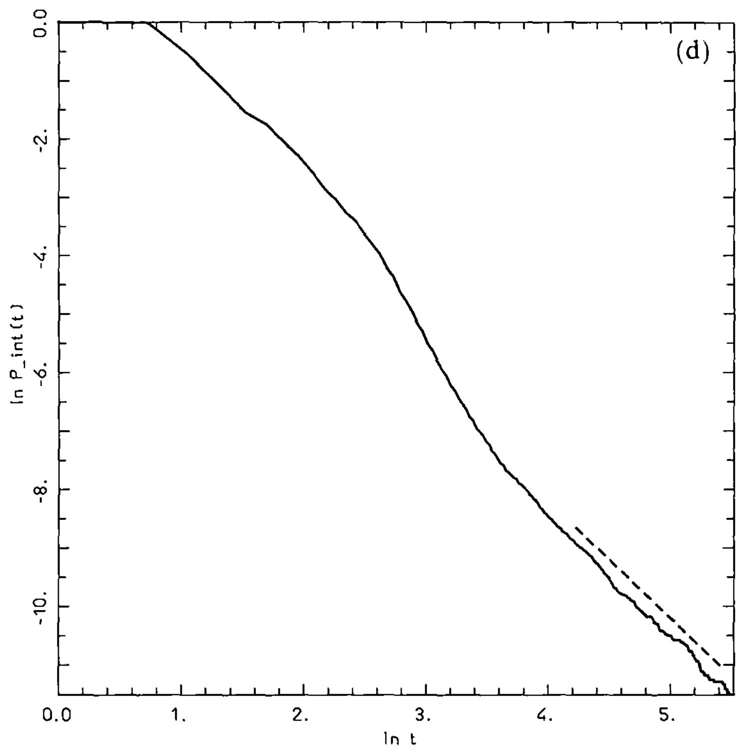

Fig. 3 (*continued*)

$\sum_{k=\tau}^{\infty} n_c^{-1}(k-\tau)\,\wp(k)$ and this sum is easily seen to coincide with $\sum_{k=\tau}^{\infty} n_c^{-1} P_{\rm int}(k)$.

Our first step consists in checking this proposed analogy with correlation functions by investigating critical cases where theoretical estimates predict power-law behavior. As regards D in the tangent case, our results are shown in Figs. 3a and 3b: the (discrete) collision dynamics is characterized by an n^{-2} long-time tail for $P_{\rm int}(n)$ [which corresponds to an m^{-1} asymptotic behavior for C_m, in accord with ref. 3 cf. (5)], while the continuous-time $P_{\rm int}(t)$ exhibits an exponential decay[2] [cf. (6)]. The opposite situation is observed when dealing with the ∞-H case (Figs. 3c and 3d): the continuous-time dynamics is characterized by asymptotic power-law decay [$P_{\rm int}(t) \sim t^{-2}$, $C_\tau \sim \tau^{-1}$] in accord with ref. 11 [cf. (7)], while the collision dynamics is characterized by exponential decay.[10] The plots have been obtained by iterating a single initial condition: the resulting profiles are robust with respect to varying initial conditions or averaging over a small number of different trajectories, as in ref. 28.

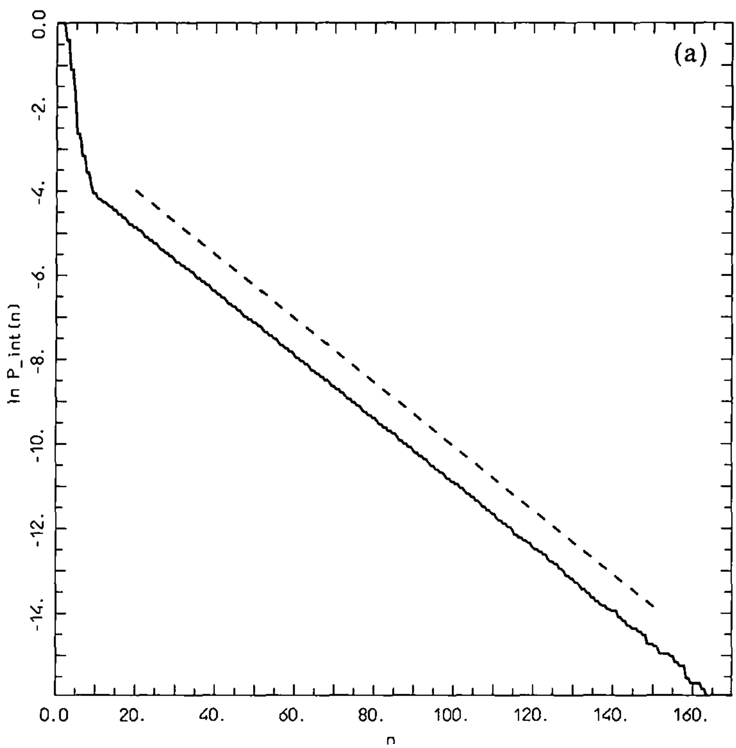

Fig. 4. (a) ln $P_{\rm int}(n)$ vs. n for a hyperbolic D case ($R = 20.518$, $N_{\rm cross} = 10^8$); the dashed line is determined by a least square fit. (b) ln $P_{\rm int}(t)$ vs. t for the same D case; again the straight dashed line results from a least square fit.

The ability to reproduce critical behavior suggests that the same method might be applied to explore hyperbolic parameter regions: we thus considered a number of D cases. A typical outcome is plotted in Figs. 4a and 4b, (here t is the absolute time; $xm = 1$ throughout all D investigations). By considering all points after the initial transient, an unweighted least-square fit gives for the slopes $\gamma_t = 6.550 \times 10^{-2} \pm 2 \times 10^{-5}$ and $\gamma_n = 7.563 \times 10^{-2} \pm 6 \times 10^{-5}$ [notation is as follows: $P_{\rm int}(t) \sim \exp(-\gamma_t \cdot t)$, $P_{\rm int}(n) \sim \exp(-\gamma_n \cdot n)$].

In Fig. 5 we show how these exponents vary by changing R in the D case: a smooth dependence is exhibited, roughly according to a power-law for large radius, within the hyperbolic range: in all of our simulations no deviation from pure exponential decay is observed.

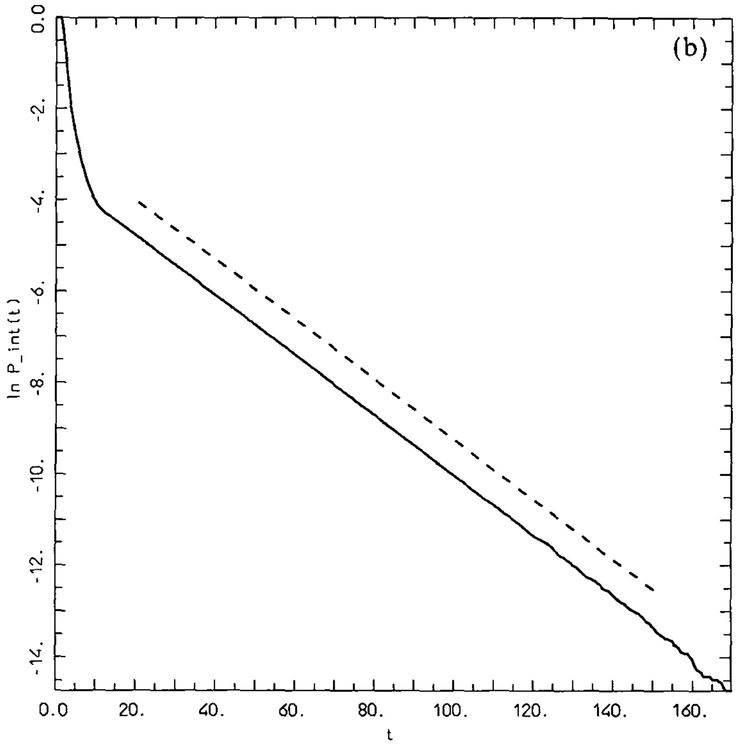

Fig. 4 (*continued*)

3. CORRELATION FUNCTIONS

We start by investigating velocity–velocity correlation functions: for the D case we consider components of the velocity along a direction which is diagonal with respect to the orientation of Fig. 1 (this is equivalent to the vertical direction in the notation of ref. 7), while we consider components along the vertical direction in S systems (Fig. 2): we denote these components by $\sharp$. Correlation functions (continuous time) are thus denoted by

$$\mathscr{C}(t) = \langle v_\sharp(t)\, v_\sharp(0) \rangle$$

Angular brackets indicate averages over the invariant measure. For (discrete time) collision dynamics we consider analogously

$$\mathscr{C}(n) = \langle v_\sharp(n)\, v_\sharp(0) \rangle$$

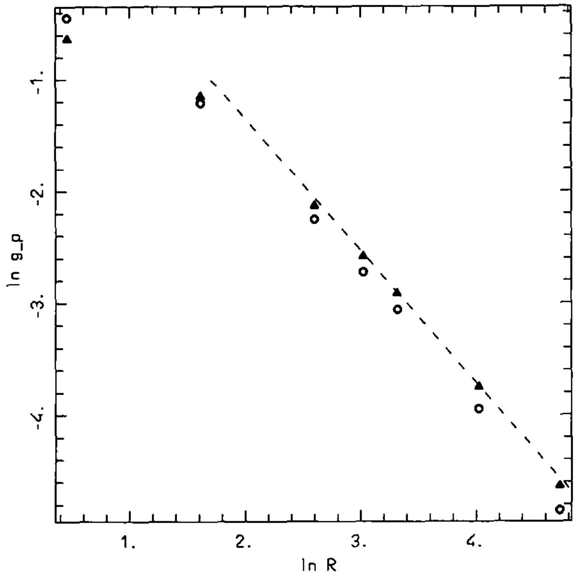

Fig. 5. $\ln \gamma_n(\triangle)$ and $\ln \gamma_t(\bigcirc)$ vs. $\ln R$: errors are within the symbol heights, the dashed line has a slope -1.18.

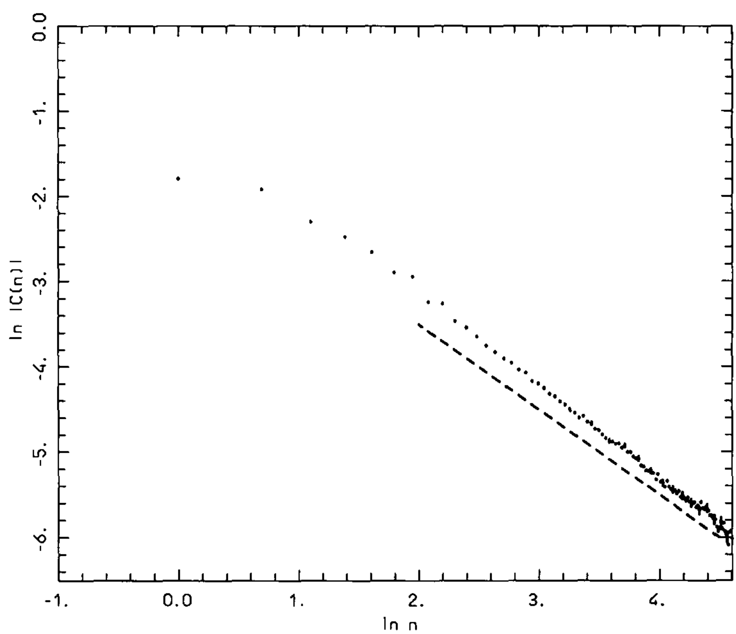

Fig. 6. $\ln |C(n)|$ vs. $\ln n$ for the D tangent case; the dashed line has a slope -1. The least square fit over all available data gives as power-law exponent -1.05 ± 0.02.

We will always evaluate numerically these functions by Monte Carlo integration over a set of N_{ph} initial conditions. The random generator we use in performing the integration is based on a subtractive method (suggested by Knuth[31]): occasionally we checked that linear congruential methods did not alter the results (even the use of a regular grid of initial conditions does not lead to discrepancies; see ref. 6). As a further check we reproduced the results of ref. 7 with our methods.

As a warmup we again consider the (discrete dynamics) tangent case for D systems: results are reported in Fig. 6 (which refers to $N_{ph} = 10^7$), in which a power-law behavior is clearly exhibited.

For the D "standard"[5,7] parameter value ($R = 2.236...$) we reobtained the data shown by ref. 7 for $\mathscr{C}(t) = \langle v_{\#}(t)\, v_{\#}(0)\rangle$: for a number of initial conditions $N_{ph} = 10^7$ we get $\gamma_v = 0.56 \pm 0.02$ (by fitting the maxima not spoiled by statistical errors: note that our time scale differs from the one considered in ref. 7 by a factor $4 \cdot \sqrt{2}$ due to a different choice of the overall length scale). As before, γ_v indicates the exponential decay rate $[\mathscr{C}(t) \sim \exp(-\gamma_v \cdot t)]$.

In the standard case the decay is so rapid that extending the range of confidence by including significantly more (statistical error free) maxima is hopeless: we then investigated a number of other D cases in which the value of R is increased. Obviously in this case also initial transients are expanded, as the system is "less hyperbolic," but nevertheless we are able to generate much longer sequences of significant maxima (due to fast decrease of the exponential rate of decay). The duration of transients is supposed to be related to the inverse Lyapunov exponent, which for large radius scales like $\lambda_R \approx C \cdot R^{-1/2}$[32]: as a matter of fact,[9]

$$\lambda_R = v \cdot \lim_{t \to \infty} \frac{1}{t} \int_0^t d\tau\, \kappa_R(x_\tau) \tag{11}$$

where

$$\kappa_R(x) = \cfrac{1}{l_0 + \cfrac{1}{-\cfrac{2}{R\cos\phi_1} + \cfrac{1}{l_1 + \cfrac{1}{-\cfrac{2}{R\cos\phi_2} + \cdots}}}} \tag{12}$$

$\{l_i\}$ is the sequence of distances between successive past collisions and $\{\phi_i\}$ denotes the angles of outgoing $\mathbf{v}_{out}$ with respect to the outward normal.

Table I. Half Periods of Oscillation for a Number of D Cases[a]

R	$t_p/2$
5.00	1.42 ± 0.02
7.81	1.41 ± 0.05
10.63	1.41 ± 0.03
13.45	1.41 ± 0.06
14.87	1.41 ± 0.03
20.52	1.41 ± 0.06
27.59	1.41 ± 0.06

[a] Errors are within *a priori* bounds in terms of the unit step used in numerical simulation.

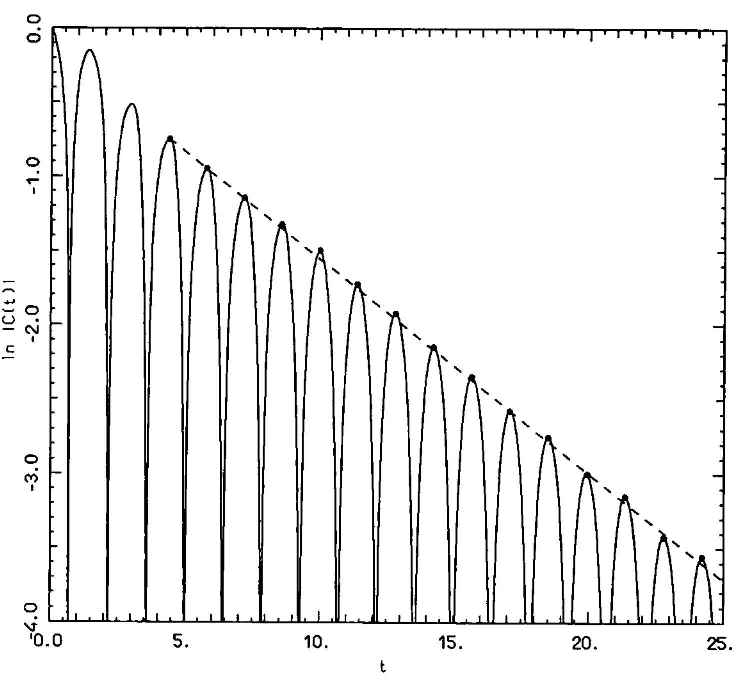

Fig. 7. ln $|C(t)|$ vs. t for a D case ($R=5$, $N_{ph}=5\times10^6$); statistical errors relative to maxima are within circles; the dashed line is a least square fit using maximum points.

The expression (11) may be evaluated explicitly in a number of cases: for the shortest periodic orbit we get

$$\lambda_{\text{s.p.o.}} = \frac{v}{D-2R} \ln \left(\frac{D-R+(D^2-2DR)^{1/2}}{R} \right) \tag{13}$$

(D is the distance between facing disks), which has the correct asymptotic behavior: in all examined cases we checked that the numerical value of the Lyapunov exponent is quite close to $\lambda_{\text{s.p.o.}}$, so we use this value as an indication of the degree of hyperbolicity, as it can be computed analytically: the approximate identification $\lambda \approx \lambda_{\text{s.p.o.}}$ is also consistent with the numerical investigations of Benettin.[32]

We studied a number of different D systems (R ranging from 1.58 to 27.59): the structure of $\mathscr{C}(t)$ is always the same: it exhibits pure exponential decay and a superimposed "universal" oscillatory behavior (there is a regular alternation of maxima and minima: the half periods are reported in Table I for a number of cases in which we have more than 15 significant

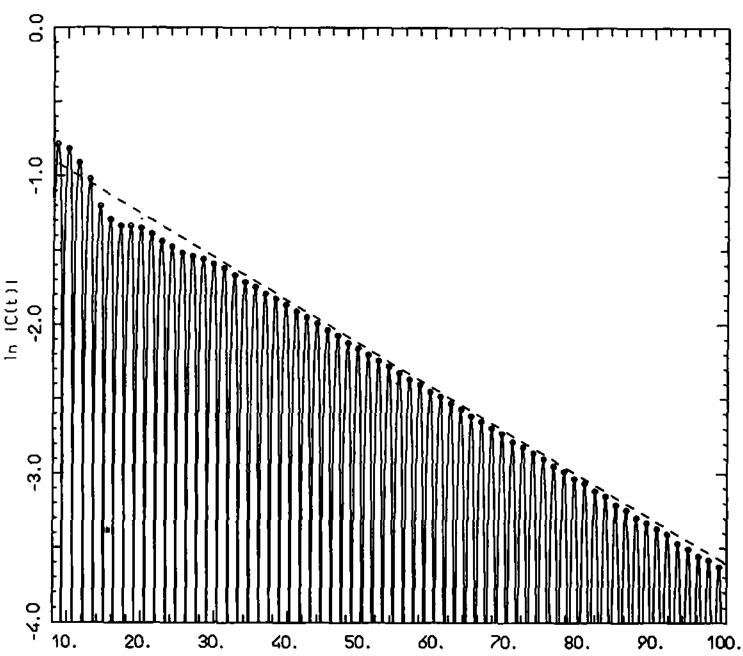

Fig. 8. ln $|C(t)|$ vs. t for a D case ($R = 14.866$, $N_{\text{ph}} = 10^7$); the dashed line is a least square fit using maximum points; statistical errors at maxima are within circles.

maxima and minima): the periods of oscillation are always compatible with the limiting value $(2 \cdot \sqrt{2})$ for $R \to \infty$.[6] Some of the results are plotted in Figs. 7 and 8.

A few comments are due, in order to appreciate better the limit of validity of our simulations: in all cases we studied, the location of maxima is well within the bound imposed by statistical error [whose order of magnitude is, however, estimated heuristically, using (8)]. Other potential sources of error[7, 20–22] involve an estimate of the initial transient and errors implied by the exponential divergence of trajectories. In particular, in ref. 7 it has been remarked (as we already recalled in the introduction) that after a time $T_{\max}$ (such that $\varepsilon \geqslant [\exp(\lambda T_{\max})]/d_m$, where d_m is the machine precision -10^{-16} for double-precision calculations and ε is the statistical error) the phase averages employed in correlation function calculations are to be interpreted as *hydrodynamic* averages, as we can no longer claim that we are following real trajectories of the system. This is a subtle issue and we have no theoretical breakthrough (such as extension of shadowing properties) to put forward: empirically it is true, however, that D systems seem robust with respect to this issue, and breaking the $T_{\max}$

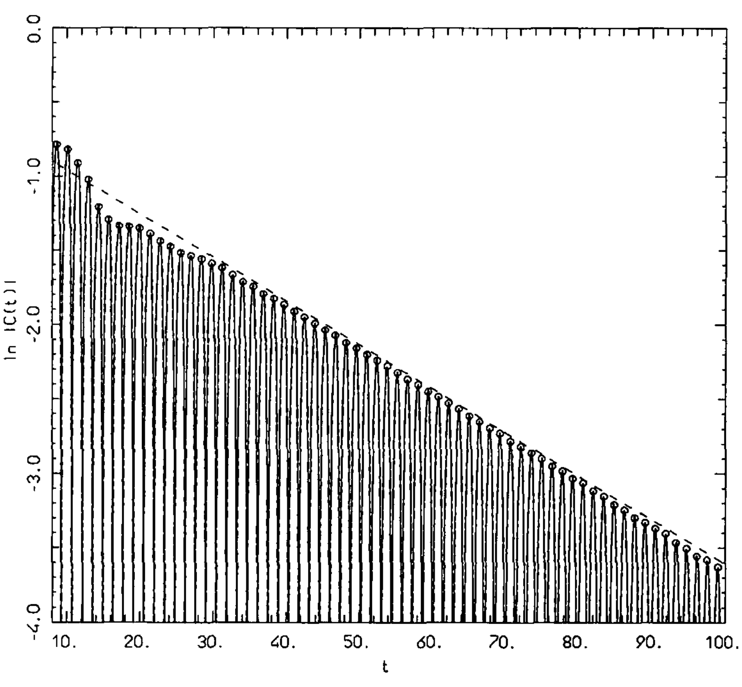

Fig. 9. Same as Fig. 8: the almost unobservable dashed-dotted line is obtained with $N_{\mathrm{ph}} = 10^7$, but using single-precision arithmetic, for which $T_{\max} \simeq 33.5$.

Table II. Estimates of Transients and Maximum Dynamical Error-Free Correlation Times[a]

R	N_{ph}	$2/\lambda_{s.p.o.}$	T_{max}
5.000	5×10^6	3.7	53.9
14.866	10^7	6.5	92.8
2.236	2.5×10^7	2.4	34.1
20.518	2×10^7	7.7	109.4

[a] The values in the last column refer to double-precision computations. The data refer to Figs. 7, 8, 12, and 13, respectively.

barrier does not produce any sensible change in the estimate of slopes (see Fig. 9). In the absence of any good rigorous argument, our plots (and γ_v estimates) refer, however, to simulations within the former limit (the relevant estimates are reported in Table II). No known results are available to get quantitative estimates of initial transients; these should be connected to inverse Lyapunov exponents (and the estimate $T_{\text{transient}} \approx 2/\lambda$ was empirically used in ref. 7; (see again Table II for estimates regarding our simulations): in estimating the slopes we excluded maxima within this range, but we hardly see any systematic quantitative change regularly taking place after some $t \approx C/\lambda_{s.p.o}$ initial time scale].

In Fig. 10 we show how γ_v varies with R: for large values of R a power-law behavior of exponent seems to show up: we do not have any scaling argument capable of explaining this behavior even approximately, but we believe that this is worth further investigation (maybe via some mean-field treatment, in the same spirit as ref. 32). While again a power-law behavior is obtained, the corresponding "critical exponent" is different from the former one, obtained in the context of "crossing statistics" (see Fig. 5).

Our last numerical experiment with velocity–velocity correlation functions concerns the ∞-H case: since in the D case varying geometrical parameters proved to be useful, we adopted the same point of view here, with the goal to see a clear transition to power-law decay (which was not exhibited, for instance, by the parameter choice of ref. 7). In Fig. 11 we show a case in which the transition to $1/t$ behavior seems evident; further cases will be dealt with elsewhere: a really challenging problem is whether rigorous arguments can predict the extent of the (exponential) transient behavior, which holds for quite a long time in a number of cases.

While we feel confident that the data strongly push toward the conclusion that velocity–velocity correlation functions decay in a pure exponential fashion for the D case, still we have to reexamine older attempts,[6] which led to different conclusions. So now we consider correlation functions

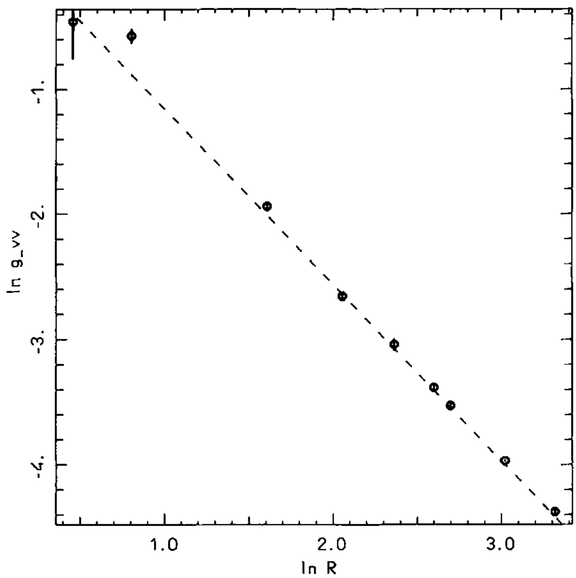

Fig. 10. ln γ_v vs. ln R for various D cases; errors are marked by vertical bars; the slope of the dashed line is -1.41.

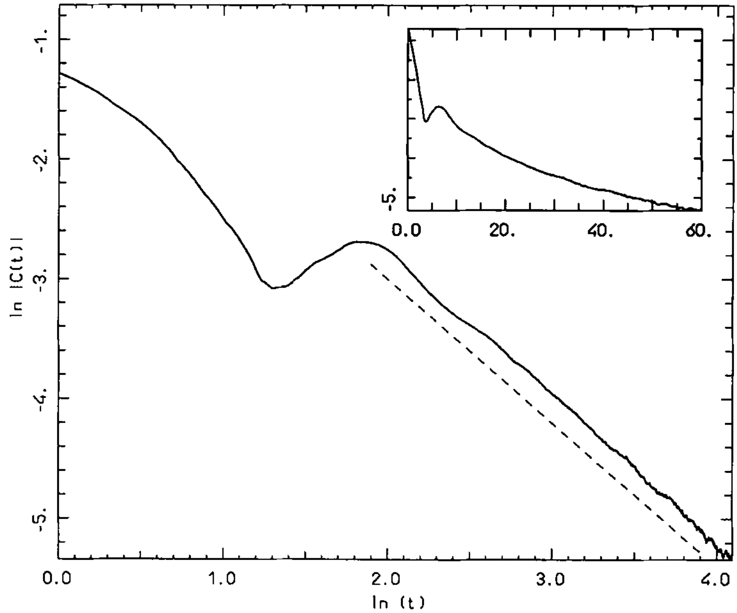

Fig. 11. ln $|C(t)|$ vs. ln t for an ∞-H case ($xm = 0.5$, $r_1 = r_2 = 0.09$, $N_{ph} = 10^7$); the dashed line has a slope -1.2; the inset represents the same data points plotted as ln $|C(t)|$ vs. t.

involving characteristic functions of some subset of the phase space: if $\mathscr{A} \subset \mathscr{M}$, the corresponding correlation function is defined as

$$\mathscr{C}_{\mathscr{A}}(t) = \left| \frac{\langle \chi_{\mathscr{A}}(t) \cdot \chi_{\mathscr{A}}(0) \rangle_{\mathscr{A}} - Vol(\mathscr{A})}{1 - Vol(\mathscr{A})} \right| \tag{14}$$

where $\langle \ldots \rangle_{\mathscr{A}}$ denotes a phase average, ruled by the invariant measure, with initial conditions belonging to $\mathscr{A}$. We remark, in the spirit of the ideas mentioned in the introduction, that here we are treating dynamical variables of a much less smooth character, so *a priori* we do not expect the same features governing correlation decay. By inspecting Figs. 12 and 13 we see that many of the regularity features we reported upon are lost: the oscillations look much less regular than in the former case (and this is connected to symmetry loss in the choice of $\mathscr{A}$), but, more strikingly, no clear pure exponential decay is exhibited (while the pattern of maxima is apparently compatible with subexponential behavior; see Fig. 13). While this behavior might be due to abnormal transients [even if the decay seems robust with variation of the ratio $vol(\mathscr{A})/vol(\mathscr{M})$, which is of the order of

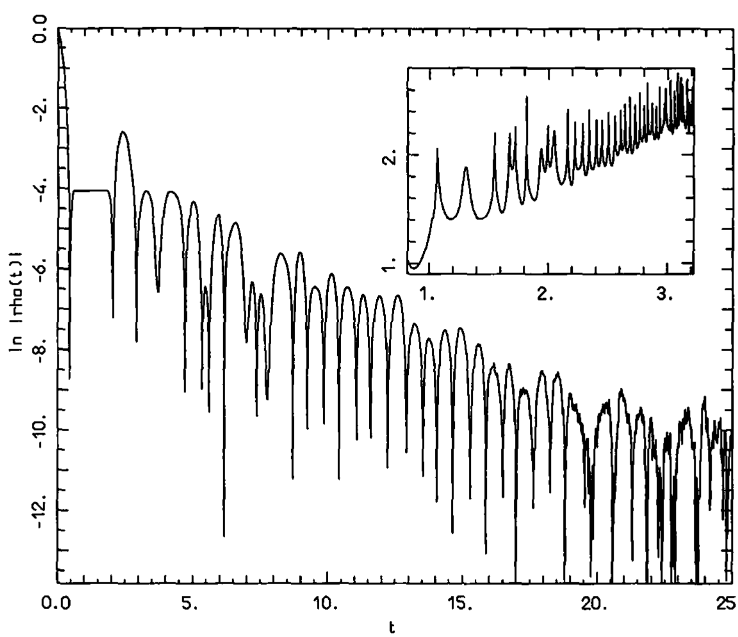

Fig. 12. $\ln \mathscr{C}_{\mathscr{A}}(t)$ vs. t for a D case ($R = 2.236$, $N_{\text{ph}} = 2.5 \times 10^7$, $x \in [0, 0.25]$, $y \in [0, 0.25]$, $\theta_v \in [0.1, 1.1]$); the inset represents the same data, but plotted as $\ln |\ln C(t)|$ vs. $\ln t$; a stretched exponential fit on nonnoisy minima gives $\gamma_{\text{se}} = 0.531 \pm 0.002$ [cf. (6)].

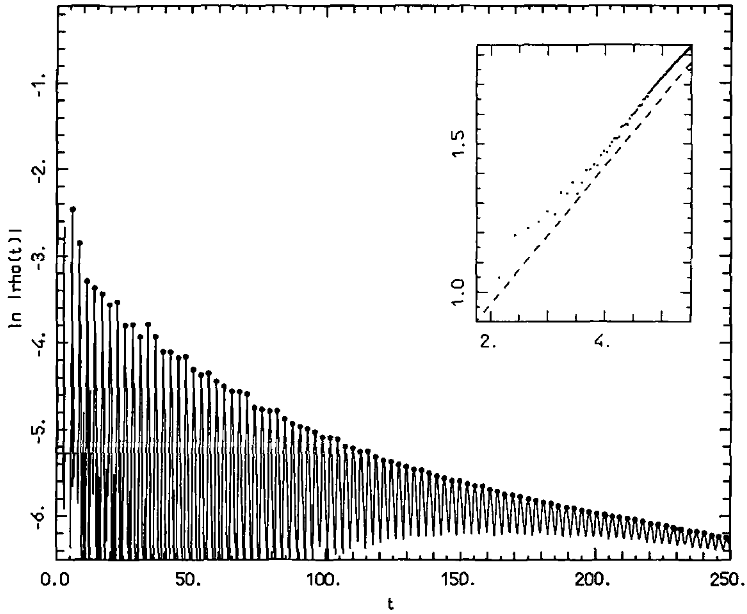

Fig. 13. $\ln \mathscr{C}_{\mathscr{A}}(t)$ vs. t for a D case ($R = 20.518$, $N_{\text{ph}} = 2 \times 10^7$, $x \in [0, 0.25]$, $y \in [0, 0.25]$, $\theta_v \in [0.1, 1.1]$); the inset represents the maxima (indicated by $\circ$ in the main graph), but plotted as $\ln |\ln C_{\max}(t)|$ vs. t; a stretched exponential fit on these points gives $\gamma_{\text{se}} = 0.232 \pm 0.002$ [cf. (6)].

1/50 in our computations], we remark that it is also compatible with the view that correlations are very sensitive to the degree of smoothness of the dynamical functions employed, with loss of smoothness inducing slower decays.

4. CONCLUSIONS

Though the problem of correlation decay has long attracted the attention of the dynamical system community, many issues are still unresolved and much effort is still being devoted to gaining a better understanding. Here we have addressed the problem of numerical investigations on dynamical systems with singularities: through a number of techniques the theoretical expectations for marginally stable situations have been reproduced (even though it would be important to have *a priori* rigorous estimates of initial transients). When the system is hyperbolic (still retaining singularities), velocity–velocity correlation functions are shown to exhibit pure exponential decay: moreover, the decay rates seem to scale regularly with variations of the geometrical parameter: similar scaling relations seem to hold for survival probabilities. This is apparently in accord with recent rigorous results[18, 8] as well as numerical experiments in ref. 7. Correlation functions involving less smooth phase functions exhibit a more complex behavior: whether this is due to abnormal transients or indicates a qualitative change is still an unresolved issue, which we believe is relevant to a deeper understanding of this class of dynamical system.

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